How Celestial Objects Move - Note on Calculations

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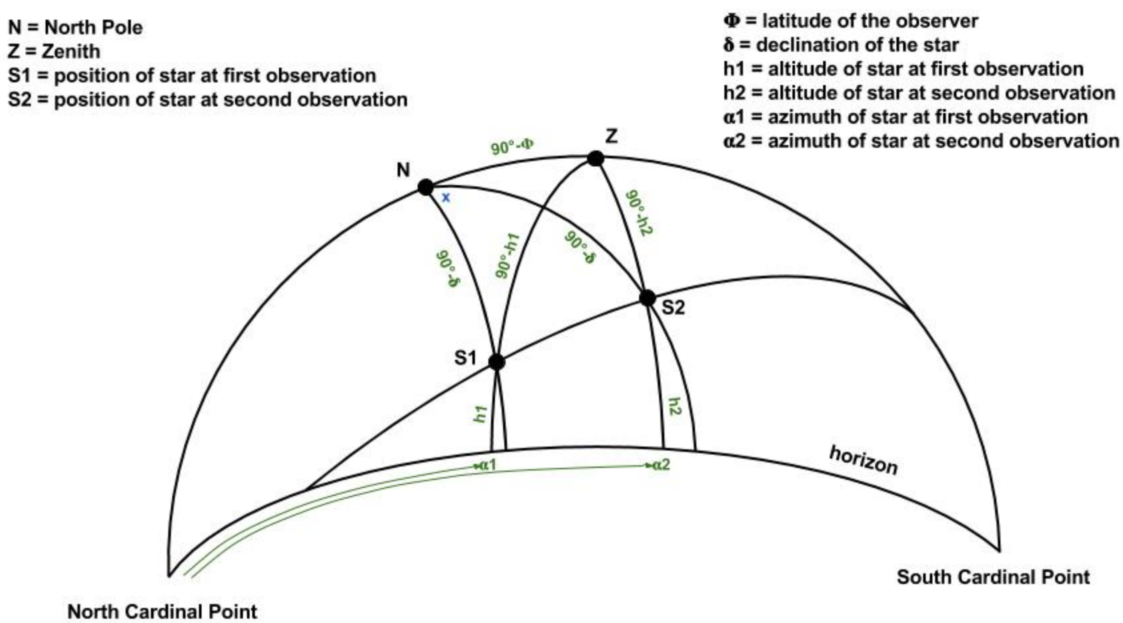

Figure 1

Our goal is to find the apparent rotation of the star around the North Pole during a period of time, given two measurements are taken at the same hour in two different days. We denote the angle of rotation by x, the measured altitudes by h_1, h_2, the measured azimuths by A_1, $A2$, the declination by δ, and the latitude by ϕ. See figure 1.

Since the declination is measured from the equator, the star will always be $90° - \delta$ from the North Pole. The latitude is the angle between the zenith and the equator, so there will be $90° - \phi$ between the zenith and the North Pole. All the circles in the drawing are "great circles" on the sphere: circles on the sphere that have their center at the center of the sphere. It is for triangles made of great circles that we can apply the spherical trigonometry formulas.

For our goal, there will be two methods from which we can derive our results, which are equivalent. It is useful to look at both because they have a different dependence on the measured quantities (the first method depends both on the azimuths and altitudes, while the second one depends only on the altitudes), so they will be affected differently by the measurement errors.

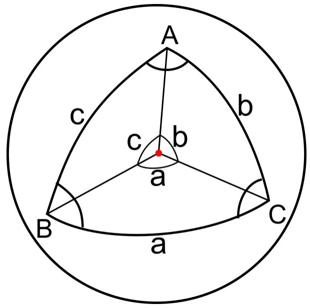

Figure 2

We will make use of a spherical trigonometry formula, called the cosine rule. Read more about spherical trigonometry on `https://en.wikipedia.org/wiki/Spherical_trigonometry` See figure 2 . The cosine rule:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

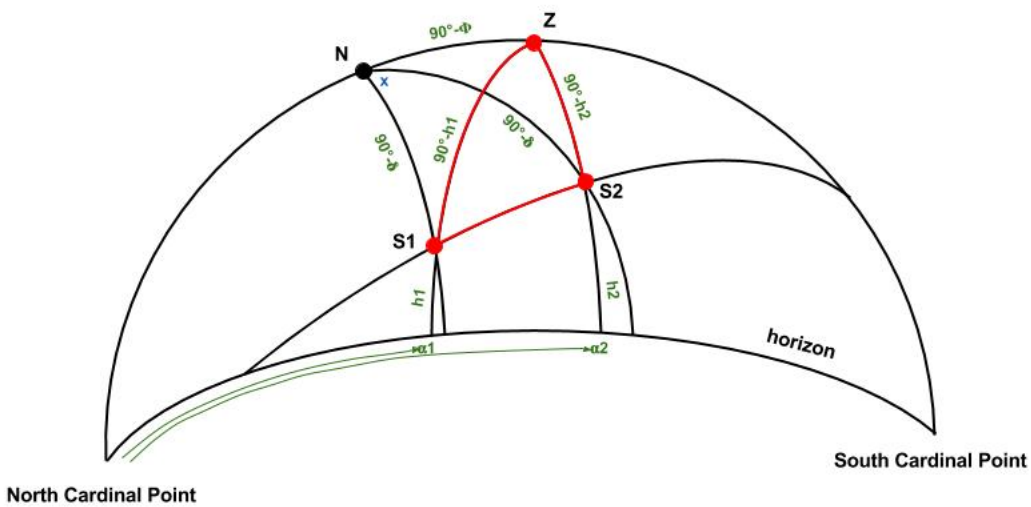

Figure 3

In the triangle S_1ZS_2 (see figure 3) we apply the cosine rule and obtain:

$$\cos \widehat{S_1S_2} = \cos(90^\circ - h_1)\cos(90^\circ - h_2) + \sin(90^\circ - h_1)\sin(90^\circ - h_2)\cos \widehat{S_1ZS_2}$$

$$\widehat{S_1ZS_2} = \alpha_2 - \alpha_1$$

$$\cos(90^\circ - \alpha) = \sin \alpha$$

$$\sin(90^\circ - \alpha) = \cos \alpha$$

$$\cos \widehat{S_1S_2} = \sin h_1 \sin h_2 + \cos h_1 \cos h_2 \cos(\alpha_2 - \alpha_1)$$

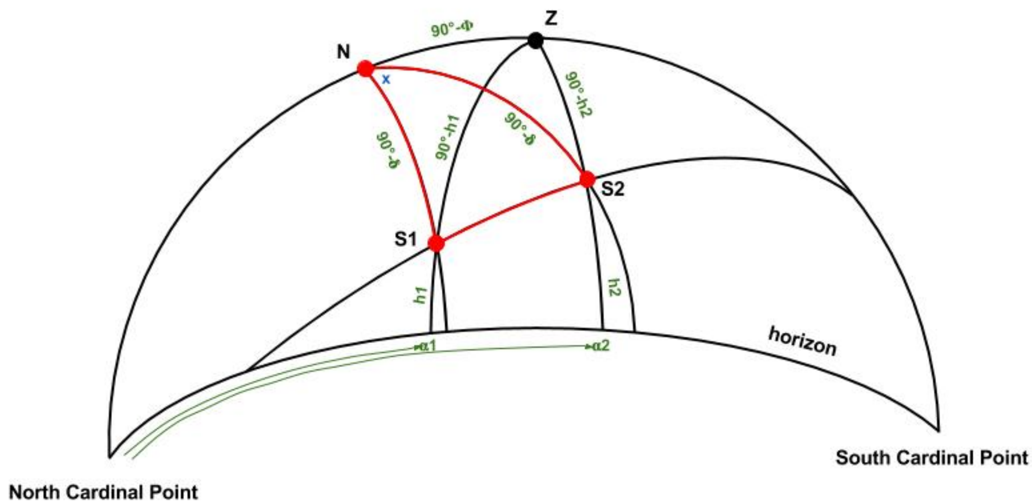

Figure 4

Method 1

In the triangle S_1NS_2 (see figure 4) we apply the cosine rule and obtain:

$$\begin{aligned}\cos\widehat{S_1S_2} &= \cos(90^\circ-\delta)\cos(90^\circ-\delta)+\sin(90^\circ-\delta)\sin(90^\circ-\delta)\cos\widehat{x}\\ &= \sin\delta\sin\delta+\cos\delta\cos\delta\cos\widehat{x}\end{aligned}$$

$$\begin{aligned}\cos\widehat{x} &= \frac{\cos\widehat{S_1S_2}-(\sin\delta)^2}{(\cos\delta)^2}\\ &= \frac{\sin h_1\sin h_2+\cos h_1\cos h_2\cos(\alpha_2-\alpha_1)-(\sin\delta)^2}{(\cos\delta)^2}\end{aligned} \tag{1}$$

$$\widehat{x} = \arccos\left(\frac{\sin h_1\sin h_2+\cos h_1\cos h_2\cos(\alpha_2-\alpha_1)-(\sin\delta)^2}{(\cos\delta)^2}\right)$$

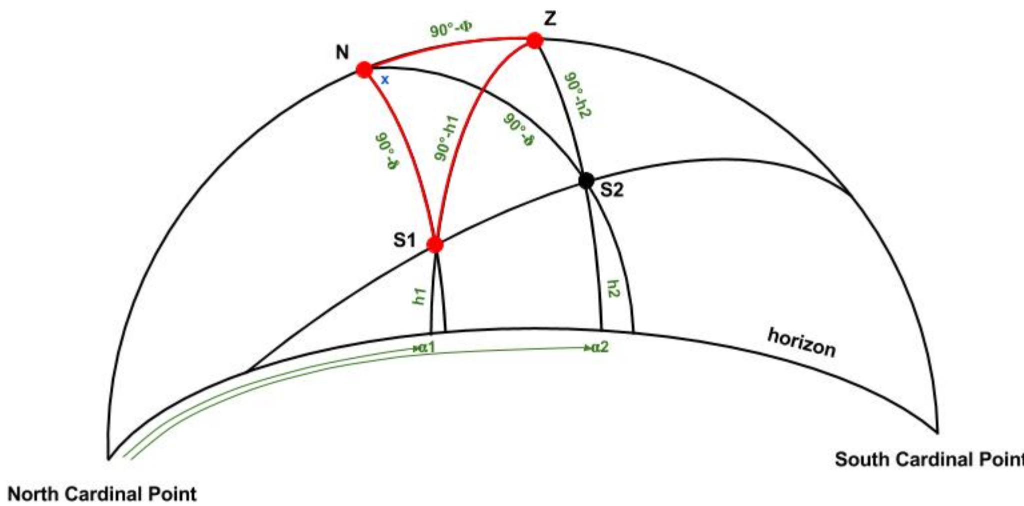

Figure 5

Method 2

In the triangle S_1NZ (see figure 5) we apply the cosine theorem and we obtain:

$$\cos(90^\circ - h_1) = \cos(90^\circ - \delta)\cos(90^\circ - \phi) + \sin(90^\circ - \delta)\sin(90^\circ - \phi)\cos\widehat{S_1NZ}$$
$$\sin h_1 = \sin\delta\sin\phi + \cos\delta\cos\phi\cos\widehat{S_1NZ}$$

$$\cos\widehat{S_1NZ} = \frac{\sin h_1 - \sin\delta\sin\phi}{\cos\delta\cos\phi} \tag{2}$$

$$\widehat{S_1NZ} = \arccos\left(\frac{\sin h_1 - \sin\delta\sin\phi}{\cos\delta\cos\phi}\right)$$

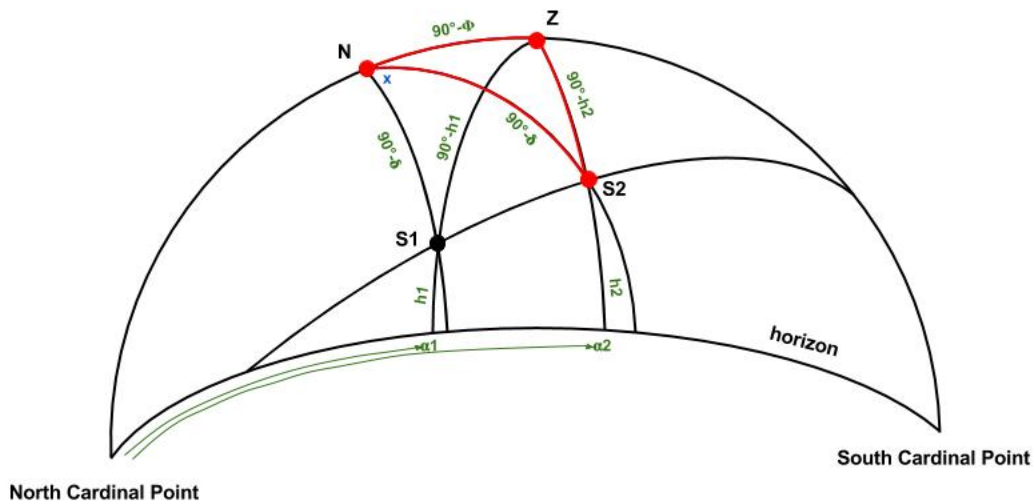

Figure 6

In the triangle S_2NZ (see figure 6) we apply the cosine theorem and we obtain:

$$\cos(90^\circ - h_2) = \cos(90^\circ - \delta)\cos(90^\circ - \phi) + \sin(90^\circ - \delta)\sin(90^\circ - \phi)\cos\widehat{S_2NZ}$$
$$\sin h_2 = \sin\delta\sin\phi + \cos\delta\cos\phi\cos\widehat{S_2NZ}$$

$$\cos\widehat{S_2NZ} = \frac{\sin h_2 - \sin\delta\sin\phi}{\cos\delta\cos\phi} \tag{3}$$

$$\widehat{S_2NZ} = \arccos\left(\frac{\sin h_2 - \sin\delta\sin\phi}{\cos\delta\cos\phi}\right)$$

$$\begin{aligned}\widehat{x} &= \widehat{S_2NZ} - \widehat{S_1NZ}\\ &= \arccos\left(\frac{\sin h_2 - \sin\delta\sin\phi}{\cos\delta\cos\phi}\right) - \arccos\left(\frac{\sin h_1 - \sin\delta\sin\phi}{\cos\delta\cos\phi}\right)\end{aligned} \tag{4}$$